POLYNOMIAL FUNCTORS OF MODULES

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Och när jag stod där gripen, kall av skräck
och fylld av ängslan inför hennes tillstånd
begynte plötsligt mimans fonoglob
att tala till mig på den dialekt
ur högre avancerad tensorlära
som hon och jag till vardags brukar mest.

— Harry Martinson, *Aniara*

Argument

We introduce the notion of *numerical functors* to generalise Eilenberg & Mac Lane's polynomial functors to arbitrary base rings. After shewing how these functors are encoded by modules over a certain ring, a characterisation of *analytic functors* is given. The article culminates in the *Polynomial Functor Theorem*, recording a precise criterion for a numerical (or polynomial) functor to admit a strict polynomial structure in the sense of Friedlander & Suslin.

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When is a polynomial functor strict polynomial? The question seems rather a fundamental and natural one; and, considering the ubiquity of polynomial functors, it will perhaps be deemed a bit surprising that nobody, as of yet, has deemed it worthy of an examination. Providing an answer, as satisfactory as one could ever hope for, is the purpose of the present note.

It will be recalled that *polynomial functors* were invented by Eilenberg & Mac Lane ([2]) in 1954, and *strict polynomial functors* by Friedlander & Suslin ([4]) in 1997 (precise definitions will follow below). As evinced by terminology, the notion of polynomial functor is weaker than that of strict polynomial functor. It will naturally be enquired: how much weaker?

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Let us recapitulate what is known of polynomial versus strict polynomial functors. For affine functors, degree 0 and 1, there will be no discernible difference between polynomial and strict polynomial functors, but they will perfectly agree. This will no longer be the case in higher degrees, as there exist polynomial functors which do not admit a strict polynomial structure. Worse, even when such a structure exists, it will usually not be unique. Example 1 below dissects a curious functor which admits several strict polynomial structures, even of different degrees.

Quadratic functors turn out to present an intermediate case, exhibiting some atypical phenomena. Example 7 dwells on this point, establishing that a quadratic integral functor F is homogeneous of degree 2 if and only if it satisfies the equation

$$F(r\alpha) = r^2 F(\alpha),$$

for any $r \in \mathbf{Q}$ and homomorphism α. In our terminology, F should be *quasi-homogeneous* of degree 2 (Definition 11). Beware, however, that a quadratic functor which is *not* quasi-homogeneous need not arise from a strict polynomial functor.

Not only will a quasi-homogeneous, quadratic functor admit a homogeneous structure; it will admit a unique such. This is singular indeed, and far from the situation in higher degrees, where there is reason to expect neither existence nor uniqueness of a strict polynomial structure, even on a functor which is *a priori* quasi-homogeneous.

Polynomial functors were initially conceived for abelian groups. While the notion, as such, is perfectly sensible for modules over any ring, it will clearly be deficient, as scalar multiplication is nowhere taken into account. As a remedy, we introduce the notion of *numerical functors* (Definition 10), designed to make sense for any binomial (or numerical) base ring. We explore their elementary properties, examine *analytic functors* (Definition 12) in some detail, and then exhibit a projective generator for the category of numerical functors, which will is found to be Morita equivalent to the category of modules over the *augmentation algebra* $\mathbf{B}[\mathbf{B}^{n\times n}]_n$, $\mathbf{B}$ betokening the base ring (Theorem 15). Morally, all theorems valid for (integral) polynomial functors will remain valid, *mutatis mutandis*, for numerical functors.

True homogeneous (strict polynomial) functors, on the other hand, are encoded by modules over the *Schur algebra* $\Gamma^n(\mathbf{B}^{n\times n})$ (Theorem 5). These two rings are linked by the *divided power map*

$$\gamma_n \colon \mathbf{B}[\mathbf{B}^{n\times n}]_n \to \Gamma^n(\mathbf{B}^{n\times n}), \quad [\sigma] \mapsto \sigma^{[n]};$$

which, by restriction of scalars, gives rise to a functor $\mathfrak{Hom}_n \to \mathfrak{Num}_n$ from homogeneous to numerical functors. Asserts Theorem 22 below:

Theorem. — *The divided power map*

$$\gamma_n \colon \mathbf{B}[\mathbf{B}^{n\times n}]_n \to \Gamma^n(\mathbf{B}^{n\times n})$$

begets the forgetful functor

$$\mathfrak{Hom}_n \to \mathfrak{Num}_n.$$

The reader will note carefully that the divided power map is not surjective, or even an epimorphism of rings, except in degrees 0, 1 and 2. This accounts for the anomalous behaviour of low-degree functors, as explained above.

After shewing how quasi-homogeneous functors, as $\mathbf{B}[\mathbf{B}^{n\times n}]_n$-modules, correspond to $\operatorname{Im}\gamma_n$-modules, we construct a section of the divided power map (Theorem 19):

$$\varepsilon_n\colon \Gamma^n(\mathbf{B}^{n\times n}) \to \mathbf{Q}\otimes_{\mathbf{Z}} \mathbf{B}[\mathbf{B}^{n\times n}]_n.$$

The grand dénouement will be the *Polynomial Functor Theorem*, Theorem 23 below:

Theorem. — *Let F be a quasi-homogeneous functor of degree n, corresponding to the $\mathbf{B}[\mathbf{B}^{n\times n}]_n$-module M and the $\operatorname{Im}\gamma_n$-module N. The following constructs are equivalent:*

A. Imposing the structure of homogeneous functor, of degree n, upon F.

B. Giving M the structure of $\operatorname{Im}\varepsilon_n$-module.

C. Giving N the structure of $\Gamma^n(\mathbf{B}^{n\times n})$-module.

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§0. Polynomial and Strict Polynomial Functors

For the entirety of this article, $\mathbf{B}$ shall denote a fixed base ring of scalars, assumed to be *binomial*[1] in the sense of Hall ([5]); that is, commutative, unital, and in the possession of binomial coëfficients. Examples include the ring of integers, as well as all $\mathbf{Q}$-algebras.

All modules, homomorphisms, and tensor products shall be taken over this $\mathbf{B}$, unless otherwise stated. We let $\mathfrak{Mod} = {}_{\mathbf{B}}\mathfrak{Mod}$ denote the category of (unital) modules over this ring.

Let $\mathfrak{XMod}$ be the category[2] of those modules that are finitely generated and free. A **module functor** is a functor

$$\mathfrak{XMod} \to \mathfrak{Mod}$$

— and, so as to avoid any misunderstandings, we duly emphasise that linearity will *not* be assumed.

We shall be wholly content to consider such restricted functors exclusively. Not only is this following tradition, but a functor defined on the subcategory

[1]This is a *numerical ring* in the terminology [3] of Ekedahl. For the equivalence of the two notions, a proof is offered in [9].

[2]The letter X herein is intended to suggest "eXtra nice modules".

$\mathfrak{X}\mathfrak{Mod}$ always has a canonical well-behaved extension to the whole module category $\mathfrak{Mod}$, as we presently expand upon.

First, let us recall what it means for a functor, not necessarily additive, to be right-exact in the sense of Bouc [1].

Definition 1. — A functor F betwixt abelian categories is **right-exact** if for any exact sequence

$$A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0\,,$$

the associated sequence

$$F(A \oplus B) \xrightarrow[-F(1_B)]{F(\alpha+1_B)} F(B) \xrightarrow{F(\beta)} F(C) \longrightarrow 0$$

is also exact. ◇

This definition agrees with the usual one in the case of an additive functor. In fact, the usual definition actually *implies* additivity of the functor, which renders it useless for our purposes.

Theorem 1 ([1], Theorem 2.14). — *Any functor $\mathfrak{X}\mathfrak{Mod} \to \mathfrak{Mod}$ has a unique extension to a functor $\mathfrak{Mod} \to \mathfrak{Mod}$ which is right-exact and commutes with inductive limits.*

No serious imposition shall thus result from considering only the said restricted functors $\mathfrak{X}\mathfrak{Mod} \to \mathfrak{Mod}$, as will be done henceforth.

Let us now bring to mind the classical notions of polynomiality. The subsequent definitions made their first appearance in print, albeit somewhat implicitly, in Eilenberg & Mac Lane's monumental article [2], sections 8 and 9:

Definition 2. — Let $\varphi\colon M \to N$ be a map of modules. The nth **deviation** of φ is the map

$$\varphi(x_1 \diamond \cdots \diamond x_{n+1}) = \sum_{I \subseteq [n+1]} (-1)^{n+1-|I|} \varphi\left(\sum_{i \in I} x_i\right)$$

of $n+1$ variables. ◇

Definition 3. — The map $\varphi\colon M \to N$ is **polynomial** of degree n if its nth deviation vanishes:

$$\varphi(x_1 \diamond \cdots \diamond x_{n+1}) = 0$$

for any $x_1, \ldots, x_{n+1} \in M$. ◇

Definition 4. — The functor $F\colon \mathfrak{X}\mathfrak{Mod} \to \mathfrak{Mod}$ is said to be **polynomial** of degree (at most) n if every arrow map

$$F\colon \operatorname{Hom}(M,N) \to \operatorname{Hom}(F(M),F(N))$$

is. ◇

We now recall the strict polynomial maps ("lois polynomes") from the work of Roby and the strict polynomial functors introduced by Friedlander and Suslin. The base ring **B** may here be taken commutative and unital only.

Definition 5 ([7], section 1.2). — A **strict polynomial map** is a natural transformation

$$\varphi\colon M \otimes - \to N \otimes -$$

betwixt functors $\mathfrak{CAlg} \to \mathfrak{Set}$, where $\mathfrak{CAlg} = {}_{\mathbf{B}}\mathfrak{CAlg}$ designates the category of commutative, unital algebras over the ring **B**, and $\mathfrak{Set}$ denotes the category of sets. ◇

Strict polynomial maps decompose as the direct sum of their homogeneous components ([7], Proposition I.4).

Definition 6 ([4], Definition 2.1). — The functor $F\colon \mathfrak{XMod} \to \mathfrak{Mod}$ is said to be **strict polynomial** of degree n if the arrow maps

$$F\colon \operatorname{Hom}(M,N) \to \operatorname{Hom}(F(M),F(N))$$

have been given a (multiplicative) strict polynomial structure. ◇

The following (slightly paraphrased) result is taken from Salomonsson's investigations of strict polynomial functors, stated merely for the purpose of later comparison with the numerical case.

Theorem 2 ([8], Propositions 2.3, 2.5). — *Consider the following constructs, where A ranges over all commutative, unital algebras:*

A. *A family of ordinary functors $E_A\colon {}_A\mathfrak{XMod} \to {}_A\mathfrak{Mod}$, commuting with extension of scalars.*

B. *A functor $J\colon \mathfrak{XMod} \to \mathfrak{Mod}$ with arrow maps*

$$J_A\colon \operatorname{Hom}_A(A \otimes M, A \otimes N) \to \operatorname{Hom}_A(A \otimes J(M), A \otimes J(N)),$$

multiplicative and natural in A.

C. *A functor $F\colon \mathfrak{XMod} \to \mathfrak{Mod}$ with arrow maps*

$$F_A\colon A \otimes \operatorname{Hom}_{\mathbf{B}}(M,N) \to A \otimes \operatorname{Hom}_{\mathbf{B}}(F(M),F(N)),$$

multiplicative and natural in A (the definition of strict polynomial functor).

Constructs A and B are equivalent, but weaker than C. If, in addition, the arrow maps are presumed to be strict polynomial of some (uniformly) bounded degree, all three are equivalent.

We thus obtain the following hierarchy of functors.

- *Strict polynomial functors*, as defined previously, have bounded degree and satisfy all three conditions A, B and C.

- A functor satisfying condition C, but with no assumption on the degree, will be called **locally strict polynomial**.
- A functor satisfying the weaker conditions A and B, again without any assumption on the degree, will be called **strict analytic**.

We have found no explicit reference for the subsequent illation, but we daresay it is rather well known. It is intended to be contrasted with Theorem 11 below.

Theorem 3. — *The strict analytic functors are precisely the direct sums (or, equivalently, inductive limits) of strict polynomial functors.*

We remark that, comparable to the situation for maps, *strict polynomial functors are not determined by their underlying functors*. The strict structure constitutes auxiliary data, which may be supplied in more than one way (or possibly none at all). The example below should serve as a warning.

Example 1. — Let $\mathbf{B} = \mathbf{Z}$, let A be a commutative $\mathbf{Z}$-algebra, and let p be a prime. The ring A/pA is a bimodule over $\mathbf{Z}$ in the usual way. Keeping the left module structure, equip it with another right module structure, mediated by the Frobenius map:

$$(x + pA) \cdot a = a^p x + pA.$$

That this is a module action is a consequence of Fermat's Little Theorem. Let $(A/pA)^{(1)}$ denote the bimodule thus obtained.

Define, for any commutative algebra A, the functors

$$\begin{aligned} F_A \colon {}_A\mathfrak{XMod} &\to {}_A\mathfrak{Mod} \\ M &\mapsto A/pA \otimes M \end{aligned}$$

and

$$\begin{aligned} G_A \colon {}_A\mathfrak{XMod} &\to {}_A\mathfrak{Mod} \\ M &\mapsto (A/pA)^{(1)} \otimes M. \end{aligned}$$

These functors commute with scalar extensions; hence they give strict analytic functors.

Let $\alpha \colon M \to N$ be a homomorphism of A-modules, and let $a \in A$. As a homomorphism

$$A/pA \otimes M \to A/pA \otimes N,$$

we have

$$F_A(a\alpha) = 1 \otimes a\alpha = a \otimes \alpha = aF_A(\alpha),$$

which shews F is homogeneous of degree 1. As a homomorphism

$$(A/pA)^{(1)} \otimes M \to (A/pA)^{(1)} \otimes N,$$

we have

$$G_A(a\alpha) = 1 \otimes a\alpha = a^p \otimes \alpha = a^p G_A(\alpha),$$

which shews G is homogeneous of degree p.

None the less, when regarded as functors only, F and G are both linear, and, as it so were, isomorphic. This is again because of Fermat's Little Theorem:

$$(x + p\mathbf{Z}) \cdot a = a^p x + p\mathbf{Z} = ax + p\mathbf{Z},$$

and consequently

$$(\mathbf{Z}/p\mathbf{Z})^{(1)} \cong \mathbf{Z}/p\mathbf{Z}$$

as $\mathbf{Z}$-bimodules. △

Definition 7. — By a **natural transformation** $\eta\colon F \to G$ of strict polynomial functors, we mean a family of homomorphisms

$$\eta = (\eta_M\colon F(M) \to G(M) \mid M \in \mathfrak{X}\mathfrak{Mod}),$$

such that for any modules M and N, any algebra A, and any

$$\omega \in A \otimes \mathrm{Hom}(M, N),$$

the following diagram commutes:

$$\begin{tikzcd} A \otimes F(M) \arrow[r, "1\otimes\eta_M"] \arrow[d, "F(\omega)"'] & A \otimes G(M) \arrow[d, "G(\omega)"] \\ A \otimes F(N) \arrow[r, "1\otimes\eta_N"'] & A \otimes G(N) \end{tikzcd}$$

◇

We shall denote by

$$\mathfrak{SPol}_n$$

the abelian category of strict polynomial functors of degree n.

Next, rather than considering arbitrary strict polynomial functors, we shall usually limit our attention to homogeneous ones.

Definition 8. — The functor $F\colon \mathfrak{X}\mathfrak{Mod} \to \mathfrak{Mod}$ is said to be **homogeneous** of degree n if the arrow maps

$$F\colon\ \mathrm{Hom}(M, N) \to \mathrm{Hom}(F(M), F(N))$$

have been given a (multiplicative) homogeneous structure. ◇

The abelian category of homogeneous functors will be denoted by

$$\mathfrak{Hom}_n.$$

By virtue of the following familiar theorem, nothing essential will be lost by considering homogeneous functors only.

Theorem 4. — *A strict polynomial functor decomposes as a unique direct sum of homogeneous functors. The only possible natural transformation between homogeneous functors of different degrees is the zero transformation. Consequently,*

$$\mathfrak{SPol}_n = \bigoplus_{k=0}^{n} \mathfrak{Hom}_k.$$

The next theorem was proved by Friedlander & Suslin for finite fields, but does not seem to have been corroborated in full generality until the work of Salomonsson.

Theorem 5 ([8], Proposition 2.4). — *The* ***fundamental homogeneous functor***

$$\Gamma^n \operatorname{Hom}(\mathbf{B}^n, -)$$

is a small projective generator for $\mathfrak{Hom}_n$, through which there is a Morita equivalence

$$\mathfrak{Hom}_n \sim {}_{\Gamma^n(\mathbf{B}^{n\times n})}\mathfrak{Mod},$$

where $\Gamma^n(\mathbf{B}^{n\times n})$ carries the ***product multiplication*** $\alpha^{[n]} \star \beta^{[n]} = (\alpha\beta)^{[n]}$.

More precisely, the functor F corresponds to the abelian group $F(\mathbf{B}^n)$, with module structure given by the equation

$$\tau^{[n]} \cdot x = F(\tau)(x).$$

Let us finally evoke the notion of *numerical map*, which will be fundamental for the perusal to follow.

Definition 9 ([10], Definition 5). — The map $\varphi\colon M \to N$ is **numerical** of degree (at most) n if it satisfies the following two equations:

$$\varphi(x_1 \diamond \cdots \diamond x_{n+1}) = 0, \qquad x_1, \ldots, x_{n+1} \in M;$$
$$\varphi(rx) = \sum_{k=0}^{n} \binom{r}{k} \varphi\left(\underset{k}{\diamond} x\right), \qquad r \in \mathbf{B},\ x \in M.$$

$\diamond$

Theorem 6 ([10], Theorem 10). — *The map $\varphi\colon M \to N$ is numerical of degree n if and only if it can be extended to a degree n natural transformation*

$$\varphi\colon M \otimes_{\mathbf{B}} - \to N \otimes_{\mathbf{B}} -$$

of functors $\mathfrak{NAlg} \to \mathfrak{Set}$, where $\mathfrak{NAlg} = {}_{\mathbf{B}}\mathfrak{NAlg}$ denotes the category of numerical algebras over the ring **B**.

§1. Numerical Functors

We proffer the following definition extending Eilenberg and Mac Lane's polynomial functors to more general rings.

Definition 10. — The functor $F\colon \mathfrak{X}\mathfrak{Mod} \to \mathfrak{Mod}$ is said to be **numerical** of degree (at most) n if every arrow map

$$F\colon \operatorname{Hom}(M,N) \to \operatorname{Hom}(F(M),F(N))$$

is. ◇

Over the integers, the notions of polynomial and numerical functor may be equated, for then all polynomial maps are numerical. Every strict polynomial functor is numerical, and if the base ring $\mathbf{B}$ be a $\mathbf{Q}$-algebra, the two strains coincide. These assertions are consequences of the corresponding statements for *maps*. Confer the remarks succeeding Definition 5 in [10].

Example 2. — A functor is numerical of degree 0 if and only if it is constant. △

Example 3. — A functor is numerical of degree 1 if and only if it is affine; *id est*, the translate of a linear functor. △

Example 4. — Most notorious of the polynomial functors are no doubt the classical algebraic functors: the *tensor power* T^n, the *symmetric power* S^n, the *exterior power* Λ^n, and the *divided power* Γ^n. Of course, since the arrow maps of these functors are not only numerical, but homogeneous (strict polynomial) of degree n, they are in fact homogeneous of degree n. △

It should be borne in mind the fundamental difference between the two types of functors — numerical and strict polynomial — which is constantly at play. While numerical functors do sanction an interpretation as functors equipped with extra data (Theorem 6), exactly corresponding to how strict polynomial functors have been defined; this auxiliary structure is, in fact, an extravagance, and may be omitted at will. Fundamentally, they are ordinary functors satisfying certain equations, as just defined; hence, *a fortiori*, a numerical functor is uniquely determined by its underlying functor. This is not true for strict polynomial functors, as was pointed out above.[3]

Denote the category of numerical functors of degree n by

$$\mathfrak{Num}_n.$$

By simple algebraical considerations, it may be verified to be abelian (the case $\mathbf{B} = \mathbf{Z}$ is well known), and it is moreover closed under direct sums. We shall presently see that it possesses a small projective generator.

[3]The failure occurs already at the level of *maps*; confer Example 7 of [10].

Definition 11. — The numerical functor F is **quasi-homogeneous** of degree n if the extension functor[4]

$$F\colon \mathbf{Q} \otimes_{\mathbf{Z}} \mathfrak{X}\mathfrak{M}\mathfrak{o}\mathfrak{d} \to \mathbf{Q} \otimes_{\mathbf{Z}} \mathfrak{M}\mathfrak{o}\mathfrak{d}$$

satisfies the equation

$$F(r\alpha) = r^n F(\alpha),$$

for any $r \in \mathbf{Q} \otimes_{\mathbf{Z}} \mathbf{B}$ and homomorphism α. ◇

Being quasi-homogeneous is a necessary condition for a functor to admit a (strict polynomial) homogeneous structure. We shall later give a sufficient condition.

The category of quasi-homogeneous functors of degree n will be denoted by the symbol

$$\mathfrak{Q}\mathfrak{H}\mathfrak{o}\mathfrak{m}_n.$$

§2. The Hierarchy of Numerical Functors

In this section, we proceed to discuss *locally numerical* and *analytic* functors.

We say that a map φ, or a family of such, is **multiplicative** if

$$\varphi(z)\varphi(w) = \varphi(zw),$$

whenever z and w are entities such that the equation makes sense, and also

$$\varphi(1) = 1,$$

where the symbol 1 is to be interpreted in a natural way (usually differently on each side). An ordinary functor is the prime example of such a multiplicative family.

The following theorem should be compared with Theorem 2 above.

Theorem 7. — *Consider the following constructs, where A ranges over all numerical algebras:*

A. *A family of ordinary functors $E_A\colon {}_A\mathfrak{X}\mathfrak{M}\mathfrak{o}\mathfrak{d} \to {}_A\mathfrak{M}\mathfrak{o}\mathfrak{d}$, commuting with extension of scalars.*

B. *A functor $J\colon \mathfrak{X}\mathfrak{M}\mathfrak{o}\mathfrak{d} \to \mathfrak{M}\mathfrak{o}\mathfrak{d}$ with arrow maps*

$$J_A\colon \operatorname{Hom}_A(A \otimes M, A \otimes N) \to \operatorname{Hom}_A(A \otimes J(M), A \otimes J(N)),$$

multiplicative and natural in A.

C. *A functor $F\colon \mathfrak{X}\mathfrak{M}\mathfrak{o}\mathfrak{d} \to \mathfrak{M}\mathfrak{o}\mathfrak{d}$ with arrow maps*

$$F_A\colon A \otimes \operatorname{Hom}_{\mathbf{B}}(M, N) \to A \otimes \operatorname{Hom}_{\mathbf{B}}(F(M), F(N)),$$

multiplicative and natural in A (the definition of numerical functor).

[4] It will be recalled that binomial rings are torsion-free.

Constructs A and B are equivalent, but weaker than C. If, in addition, the arrow maps are presumed to be numerical of some (uniformly) bounded degree, all three are equivalent.

Proof. Given E, define J by

$$J(M) = E_{\mathbf{B}}(M)$$

and the diagram:

$$\begin{tikzcd}
\mathrm{Hom}_A(A \otimes M, A \otimes N) \arrow[r, "E_A"] \arrow[dr, dotted, "J"'] & \mathrm{Hom}_A(E_A(A \otimes M), E_A(A \otimes N)) \arrow[d, leftrightarrow] \\
& \mathrm{Hom}_A(A \otimes E_{\mathbf{B}}(M), A \otimes E_{\mathbf{B}}(N))
\end{tikzcd}$$

Conversely, given J, define the functors E by the equations

$$E_A(M) = A \otimes J(M)$$

and

$$\mathrm{Hom}_A(A \otimes M, A \otimes N) \xrightarrow{E_A = J} \mathrm{Hom}_A(A \otimes J(M), A \otimes J(N)).$$

Also, it is easy to define J from F; simply let

$$J(M) = F(M),$$

and use the following diagram:

$$\begin{tikzcd}
A \otimes \mathrm{Hom}_{\mathbf{B}}(M, N) \arrow[r, "F"] \arrow[d, leftrightarrow] & A \otimes \mathrm{Hom}_{\mathbf{B}}(F(M), F(N)) \arrow[d] \\
\mathrm{Hom}_A(A \otimes M, A \otimes N) \arrow[r, dotted, "J"'] & \mathrm{Hom}_A(A \otimes F(M), A \otimes F(N))
\end{tikzcd}$$

The left column in the diagram is an isomorphism as long as M and N are free.

The difficult part is defining F from J, provided that J is indeed of bounded degree n. The following proof is modelled on the corresponding argument for strict polynomial functors in [8]. Let M and N be two modules, and let A be any numerical algebra. Find a free resolution

$$\mathbf{B}^{(\lambda)} \longrightarrow \mathbf{B}^{(\kappa)} \longrightarrow J(M) \longrightarrow 0\ ,$$

and apply the contra-variant, left-exact functor

$$\mathrm{Hom}_A(A \otimes -, A \otimes J(N))$$

to obtain a commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_A(A \otimes J(M), A \otimes J(N)) \arrow[r, "\iota"] & (A \otimes J(N))^{\kappa} \arrow[r, "\sigma"] & (A \otimes J(N))^{\lambda} \\
& A \otimes \mathrm{Hom}(M,N) \arrow[u, "J"] \arrow[r, "\delta_n"'] & A \otimes \mathbf{B}[\mathrm{Hom}(M,N)]_n \arrow[u, dotted, "\zeta"'] &
\end{tikzcd}$$

The homomorphism

$$\iota J \colon A \otimes \mathrm{Hom}(M,N) \to (A \otimes J(N))^{\kappa}$$

may be split up into components

$$(\iota J)_k \colon A \otimes \mathrm{Hom}(M,N) \to A \otimes J(N),$$

for each $k \in \kappa$. Those are numerical of degree n, and will factor over δ_n via some linear ζ_k. Together they yield a linear map

$$\zeta \colon A \otimes \mathbf{B}[\mathrm{Hom}(M,N)]_n \to (A \otimes J(N))^{\kappa},$$

making the above square commute.

Now, $\sigma\zeta\delta_n = \sigma\iota J = 0$, which gives $\sigma\zeta = 0$. By the exactness of the upper row, ζ factors via some homomorphism

$$\xi \colon \mathbf{B}[\mathrm{Hom}(M,N)]_n \to \mathrm{Hom}(J(M), J(N)).$$

Because

$$\iota J = \zeta\delta_n = \iota\xi\delta_n$$

and ι is one-to-one, we also have $J = \xi\delta_n$. The following diagram will therefore commute:

$$\begin{tikzcd}
\mathrm{Hom}(J(M), J(N)) \arrow[r, "\iota"] & J(N)^{(\kappa)} \\
\mathrm{Hom}(M,N) \arrow[u, "J"] \arrow[r, "\delta_n"'] & \mathbf{B}[\mathrm{Hom}(M,N)]_n \arrow[u, "\zeta"'] \arrow[ul, dotted, "\xi"']
\end{tikzcd}$$

Since J factors over $\mathbf{B}[\mathrm{Hom}(M,N)]_n$, it is numerical of degree n, and so may be used to construct F. □

We thus obtain the following hierarchy of functors.

Definition 12.

- **Numerical functors** have bounded degree and satisfy all three conditions A, B and C. (This coincides with the previous definition.)
- A functor satisfying condition C, but with no assumption on the degree, will be called **locally numerical**.

- A functor satisfying the weaker conditions A and B, again without any assumption on the degree, will be called **analytic**.

◇

Example 5. — The classical algebraic functors T, S, Λ and Γ are all analytic, for they evidently satisfy condition A of the theorem.

Of these, only Λ is locally numerical. This is because, when $n > p$, the module

$$\Lambda^n(\mathbf{B}^p) = 0,$$

and hence, for given p and q, the map

$$\Lambda\colon \operatorname{Hom}(\mathbf{B}^p, \mathbf{B}^q) \to \operatorname{Hom}(\Lambda(\mathbf{B}^p), \Lambda(\mathbf{B}^q))$$

is numerical of degree $\max(p, q)$. △

§3. Properties of Numerical Functors

Since numerical functors allow for a more complicated rendition, it ought not to be surprising that natural transformations also satisfy a more involved condition. The theorem below exhibits obvious conformity with Definition 7.

Theorem 8. — *Let*

$$\eta = (\eta_M\colon F(M) \to G(M) \mid M \in \mathfrak{XMod})$$

be a natural transformation of numerical functors F and G. For any modules M and N, any numerical algebra A, and any

$$\omega \in A \otimes \operatorname{Hom}(M, N),$$

the following diagram commutes:

$$\begin{tikzcd} A \otimes F(M) \arrow[r, "1\otimes\eta_M"] \arrow[d, "F(\omega)"'] & A \otimes G(M) \arrow[d, "G(\omega)"] \\ A \otimes F(N) \arrow[r, "1\otimes\eta_N"'] & A \otimes G(N) \end{tikzcd}$$

Proof. Consider homomorphisms

$$\alpha_1, \ldots, \alpha_k\colon M \to N.$$

Assume that

$$F(a_1 \otimes \alpha_1 + \cdots + a_k \otimes \alpha_k) = \sum_\mu \binom{a_1}{m_1} \cdots \binom{a_k}{m_k} \otimes \beta_\mu$$

$$G(a_1 \otimes \alpha_1 + \cdots + a_k \otimes \alpha_k) = \sum_{\nu} \binom{a_1}{n_1} \cdots \binom{a_k}{n_k} \otimes \gamma_\nu;$$

for any $a_1, \ldots, a_k$ in any numerical algebra A, where we have abbreviated

$$\mu = (m_1, \ldots, m_k) \quad \text{and} \quad \nu = (n_1, \ldots, n_k).$$

The naturality of η ensures that

$$\sum_{\mu} \binom{a_1}{m_1} \cdots \binom{a_k}{m_k} \eta_N \beta_\mu = \sum_{\nu} \binom{a_1}{n_1} \cdots \binom{a_k}{n_k} \gamma_\nu \eta_M.$$

Specialise first to the case $a_2 = a_3 = \cdots = 0$, to obtain

$$\sum_{m_1} \binom{a_1}{m_1} \eta_N \beta_{(m_1, 0, \ldots)} = \sum_{n_1} \binom{a_1}{n_1} \gamma_{(n_1, 0, \ldots)} \eta_M.$$

Successively putting $a_1 = 0, 1, 2, \ldots$ leads to

$$\eta_N \beta_{(m_1, 0, \ldots)} = \gamma_{(m_1, 0, \ldots)} \eta_M$$

for all m_1. Proceeding inductively, one shews that

$$\eta_N \beta_\mu = \gamma_\mu \eta_M$$

for all μ. The commutativity of the diagram, for

$$\omega = a_1 \otimes \alpha_1 + \cdots + a_k \otimes \alpha_k,$$

is then demonstrated by the following instantiation:

$$\begin{tikzcd}
b \otimes x \arrow[r] \arrow[d] & b \otimes \eta_M(x) \arrow[d] \\
\sum_\mu \binom{a_1}{m_1} \cdots \binom{a_k}{m_k} b \otimes \beta_\mu(x) \arrow[r] & \left[\begin{array}{c} \sum_\mu \binom{a_1}{m_1} \cdots \binom{a_k}{m_k} b \otimes \eta_N \beta_\mu(x) \\ = \sum_\mu \binom{a_1}{m_1} \cdots \binom{a_k}{m_k} b \otimes \gamma_\mu \eta_M(x) \end{array}\right]
\end{tikzcd}$$

□

Next, we give some equivalent characterisations of numericality, which may perhaps be more convenient in practice.

Theorem 9. — *The following conditions are equivalent on a polynomial functor F of degree n.*

A.

$$F(r\alpha) = \sum_{k=0}^{n} \binom{r}{k} F\left(\underset{k}{\Diamond}\, \alpha\right),$$

for any scalar r and homomorphism α (the definition of numerical functor).

B.

$$F(r\alpha) = \sum_{m=0}^{n} (-1)^{n-m} \binom{r}{m} \binom{r-m-1}{n-m} F(m\alpha),$$

for any scalar r and homomorphism α.

A′.

$$F(r \cdot 1_{\mathbf{B}^n}) = \sum_{k=0}^{n} \binom{r}{k} F\left(\underset{k}{\Diamond} 1_{\mathbf{B}^n}\right),$$

for any scalar r.

B′.

$$F(r \cdot 1_{\mathbf{B}^n}) = \sum_{m=0}^{n} (-1)^{n-m} \binom{r}{m} \binom{r-m-1}{n-m} F(m \cdot 1_{\mathbf{B}^n}),$$

for any scalar r.

Proof. That A and B are equivalent follows from Theorem 7 of [10], as does the equivalence of A′ and B′. Clearly B implies B′, so there remains to establish the implication of B by B′.

Hence assume B′, and put

$$Z_m = (-1)^{n-m} \binom{r}{m} \binom{r-m-1}{n-m}.$$

In the case $q \leqslant n$, the equation

$$F(r \cdot 1_{\mathbf{B}^q}) = \sum_{m=0}^{n} Z_m F(m \cdot 1_{\mathbf{B}^q})$$

clearly holds, because $1_{\mathbf{B}^q}$ factors through $1_{\mathbf{B}^n}$.

Consider now the case $q > n$. By induction, assume the formula holds for $q-1$. Letting π_i denote the canonical projections, we calculate:

$$\begin{aligned}
F(r \cdot 1_{\mathbf{B}^q}) &= F(r\pi_1 + \cdots + r\pi_q) \\
&= -\sum_{I \subset [q]} (-1)^{q-|I|} F\left(\sum_{i \in I} r\pi_i\right) \\
&= -\sum_{I \subset [q]} (-1)^{q-|I|} \sum_{m=0}^{n} Z_m F\left(\sum_{i \in I} m\pi_i\right) \\
&= -\sum_{m=0}^{n} Z_m \sum_{I \subset [q]} (-1)^{q-|I|} F\left(\sum_{i \in I} m\pi_i\right) \\
&= \sum_{m=0}^{n} Z_m F(m\pi_1 + \cdots + m\pi_q) = \sum_{m=0}^{n} Z_m F(m \cdot 1_{\mathbf{B}^q}).
\end{aligned}$$

The third and sixth steps are because the qth deviation vanishes. This shews that the equation holds for $\mathrm{1}_{\mathbf{B}^q}$, for any q.

Finally, in the case of an arbitrary homomorphism $\alpha\colon \mathbf{B}^p \to \mathbf{B}^q$, we have

$$\begin{aligned} F(r\alpha) &= F(r \cdot \mathrm{1}_{\mathbf{B}^q})F(\alpha) \\ &= \sum_{m=0}^{n} Z_m F(m \cdot \mathrm{1}_{\mathbf{B}^q})F(\alpha) = \sum_{m=0}^{n} Z_m F(m\alpha), \end{aligned}$$

and the proof is finished. □

The following very pleasant formula is an immediate consequence of the corresponding formula for maps.

Recall that a *multi-set* is a set with repeated elements. When X is a multi-set, we shall denote by $|X|$ its *cardinality*, that is, the number of elements counted with multiplicity, and by $\#X$ the underlying set, called its *support*.

Theorem 10. — *The module functor F is numerical of degree n if and only if, for any scalars a_i and homomorphisms α_i, the following equation holds:*

$$F(a_1\alpha_1 \diamond \cdots \diamond a_k\alpha_k) = \sum_{\substack{\#X=[k] \\ |X|\leqslant n}} \binom{a}{X} F\left(\mathop{\Diamond}_X \alpha\right).$$

Proof. Theorem 8 of [10]. □

Example 6. — A cubic functor F is characterised by the following formulæ:

$$\begin{aligned} F(\diamond a_1\alpha_1) &= \binom{a_1}{1} F(\diamond\alpha_1) + \binom{a_1}{2} F(\alpha_1 \diamond \alpha_1) \\ &\quad + \binom{a_1}{3} F(\alpha_1 \diamond \alpha_1 \diamond \alpha_1) \\ F(a_1\alpha_1 \diamond a_2\alpha_2) &= \binom{a_1}{1}\binom{a_2}{1} F(\alpha_1 \diamond \alpha_2) + \binom{a_1}{2}\binom{a_2}{1} F(\alpha_1 \diamond \alpha_1 \diamond \alpha_2) \\ &\quad + \binom{a_1}{1}\binom{a_2}{2} F(\alpha_1 \diamond \alpha_2 \diamond \alpha_2) \\ F(a_1\alpha_1 \diamond a_2\alpha_2 \diamond a_3\alpha_3) &= \binom{a_1}{1}\binom{a_2}{1}\binom{a_3}{1} F(\alpha_1 \diamond \alpha_2 \diamond \alpha_3). \end{aligned}$$

△

§4. Analytic Functors

We now examine the analytic functors. Traditionally, analytic functors have been identified with the inductive limits of polynomial functors. The definition we gave above is no different, as we now set out to shew.

Lemma 1. — *Let F be an analytic functor and P a finitely generated, free module. Suppose $u \in F(P)$, and define the subfunctor G by*

$$G(M) = \langle F(\alpha)(u) \mid \alpha\colon P \to M \rangle .$$

Consider the natural transformation

$$\xi\colon \operatorname{Hom}(P, -) \to F,$$

given by

$$\begin{aligned} \xi_N\colon \operatorname{Hom}(P, N) &\to F(N) \\ \alpha &\mapsto F(\alpha)(u). \end{aligned}$$

If ξ_N is numerical of degree n, then so is

$$G\colon (M, N) \to \operatorname{Hom}(G(M), G(N))$$

for all M. In particular:

- *If all ξ_N are numerical, then G is locally numerical.*
- *If all ξ_N are numerical of uniformly bounded degree, then G is numerical.*

Proof. Observe that the modules $G(M)$ are invariant under the action of F. Thus, G is indeed a subfunctor of F.

Suppose ξ_N is numerical of degree n. Then, for all homomorphisms

$$\alpha, \alpha_i\colon P \to N$$

and scalars r, the following equations hold:

$$\begin{cases} F(\alpha_1 \diamond \cdots \diamond \alpha_{n+1})(u) = 0 \\ F(r\alpha)(u) = \sum_{m=0}^{n} \binom{r}{m} F\left(\Diamond_m \alpha\right)(u). \end{cases}$$

This implies that, for all homomorphisms

$$\beta, \beta_i\colon M \to N, \qquad \gamma\colon P \to M,$$

and scalars r, the following equations hold:

$$\begin{cases} F(\beta_1 \diamond \cdots \diamond \beta_{n+1}) F(\gamma)(u) = 0 \\ F(r\beta) F(\gamma)(u) = \sum_{m=0}^{n} \binom{r}{m} F\left(\Diamond_m \beta\right) F(\gamma)(u). \end{cases}$$

Hence

$$F(\beta_1 \diamond \cdots \diamond \beta_{n+1}) = 0$$

and

$$F(r\beta) = \sum_{m=0}^{n} \binom{r}{m} F\left(\underset{m}{\Diamond} \beta\right)$$

on $G(M)$, which means that every

$$G\colon (M, N) \to \operatorname{Hom}(G(M), G(N))$$

is indeed numerical of degree n. □

The following theorem should be compared with Theorem 3 above.

Theorem 11. — *The analytic functors are precisely the inductive limits of numerical functors.*

Proof. Step 1: Inductive limits of numerical, or even analytic, functors are analytic. Let the functors F_i, for $i \in I$, be analytic. For any

$$\alpha \in \mathrm{Hom}_A(A \otimes M, A \otimes N),$$

we have

$$F_i(\alpha)\colon A \otimes F_i(M) \to A \otimes F_i(N).$$

Therefore

$$\varinjlim F_i(\alpha)\colon A \otimes \varinjlim F_i(M) \to A \otimes \varinjlim F_i(N),$$

since tensor products commute with inductive limits, which yields a map

$$\varinjlim F_i\colon\ \mathrm{Hom}_A(A \otimes M, A \otimes N) \to \mathrm{Hom}_A(A \otimes \varinjlim F_i(M), A \otimes \varinjlim F_i(N)),$$

establishing that $\varinjlim F_i$ is analytic.

Step 2: Analytic functors are inductive limits of locally numerical functors. Let F be analytic. The maps

$$F\colon\ \mathrm{Hom}_A(A \otimes M, A \otimes N) \to \mathrm{Hom}_A(A \otimes F(M), A \otimes F(N))$$

are then multiplicative and natural in A. To shew F is the inductive limit of locally numerical functors, it is sufficient to construct, for any given module P and element $u \in F(P)$, a locally numerical subfunctor G of F, such that $u \in G(P)$.

To this end, define G as in the lemma:

$$G(M) = \langle F(\alpha)(u) \mid \alpha\colon P \to M \rangle.$$

Clearly $u \in G(P)$. By the lemma, G is locally numerical, if only we can shew that

$$\xi_M\colon\ \mathrm{Hom}(P, M) \to F(M)$$

is always numerical (of possibly unbounded degree).

We make use of the following fact. The module $\mathrm{Hom}(P, M)$ is finitely generated and free, because P and M are. Let $\varepsilon_1, \ldots, \varepsilon_k$ be a basis.

Let $s_1, \ldots, s_k$ be free variables, and let

$$A = \mathbf{B}\begin{pmatrix} s_1, \ldots, s_k \\ - \end{pmatrix}.$$

Since

$$F\left(\sum s_i \otimes \varepsilon_i\right) \in \mathrm{Hom}_A(A \otimes F(P), A \otimes F(M)),$$

we may write

$$F\left(\sum s_i \otimes \varepsilon_i\right)(\mathrm{I}_A \otimes u) = \sum_{|X| \leqslant n} \binom{s}{X} \otimes v_X \in A \otimes F(M)$$

for some n, and hence

$$\xi_M\left(\sum s_i \varepsilon_i\right) = F\left(\sum s_i \varepsilon_i\right)(u) = \sum_{|X| \leqslant n} \binom{s}{X} v_X.$$

Since the ε_i generate $\mathrm{Hom}(P, M)$, it follows that ξ_M is numerical of degree n.

Step 3: Locally numerical functors are inductive limits of numerical functors. Let F be locally numerical, and, given P and $u \in F(P)$, define G and ξ as before. We shall shew that G is numerical by shewing that ξ is numerical of some fixed degree.

Let $\alpha_i \colon P \to M$ be homomorphisms, let

$$B = \mathbf{B}\binom{s_1, \ldots, s_k}{-}, \qquad C = \mathbf{B}\binom{s_1, \ldots, s_k, t}{-}$$

be free numerical rings, and consider the algebra homomorphism

$$\tau \colon B \to C, \quad s_i \mapsto t s_i.$$

There is a commutative diagram:

$$\begin{tikzcd}
B \arrow[d, "\tau"'] & B \otimes \mathrm{Hom}(P, M) \arrow[r, "F"] \arrow[d, "\tau\otimes\mathrm{I}"'] & B \otimes \mathrm{Hom}(F(P), F(M)) \arrow[d, "\tau\otimes\mathrm{I}"] \\
C & C \otimes \mathrm{Hom}(P, M) \arrow[r, "F"'] & C \otimes \mathrm{Hom}(F(P), F(M))
\end{tikzcd}$$

As a consequence, we obtain, for any homomorphisms $\alpha_i \colon P \to M$:

$$\begin{tikzcd}
\sum s_i \otimes \alpha_i \arrow[r] \arrow[d] & F\left(\sum s_i \otimes \alpha_i\right) \arrow[d] \\
\sum t s_i \otimes \alpha_i \arrow[r] & \left[\begin{array}{c}(\tau \otimes \mathrm{I}) F\left(\sum s_i \otimes \alpha_i\right) \\ = F\left(\sum t s_i \otimes \alpha_i\right)\end{array}\right]
\end{tikzcd}$$

Consider now

$$F \colon \mathbf{B}\binom{s_1, \ldots, s_k}{-} \otimes \mathrm{Hom}(P, M) \to \mathbf{B}\binom{s_1, \ldots, s_k}{-} \otimes \mathrm{Hom}(F(P), F(M)),$$

and write

$$F\left(\sum s_i \otimes \alpha_i\right) = \sum_X \binom{s}{X} \otimes \beta_X,$$

for some homomorphisms $\beta_X \colon F(P) \to F(M)$.

Similarly, from contemplating

$$F \colon \mathbf{B}\binom{t}{-} \otimes \operatorname{Hom}(P,P) \to \mathbf{B}\binom{t}{-} \otimes \operatorname{Hom}(F(P),F(P)),$$

we may write

$$F(t \otimes \mathrm{1}_P) = \sum_{m \leqslant n} \binom{t}{m} \otimes \gamma_m,$$

for some number n and homomorphisms $\gamma_m \colon F(P) \to F(P)$. Observe that n is fixed, and only depends on F.

We now have

$$\begin{aligned}
\sum_X \binom{ts}{X} \otimes \beta_X &= (\tau \otimes \mathrm{1}) \left(\sum_X \binom{s}{X} \otimes \beta_X \right) = (\tau \otimes \mathrm{1}) F \left(\sum s_i \otimes \alpha_i \right) \\
&= F \left(\sum t s_i \otimes \alpha_i \right) = F \left(\sum s_i \otimes \alpha_i \right) F(t \otimes \mathrm{1}_P) \\
&= \left(\sum_X \binom{s}{X} \otimes \beta_X \right) \left(\sum_{m \leqslant n} \binom{t}{m} \otimes \gamma_m \right) \\
&= \sum_X \sum_{m \leqslant n} \binom{s}{X} \binom{t}{m} \otimes \beta_X \gamma_m.
\end{aligned}$$

The right-hand side, and therefore also the left-hand side, is of degree n in t, whence $\beta_X = 0$ when $|X| > n$.

Consequently,

$$\xi_M \left(\sum s_i \alpha_i \right) = F \left(\sum s_i \alpha_i \right) (u) = \sum_{|X| \leqslant n} \binom{s}{X} \beta_X(u),$$

and ξ is numerical of degree n. □

§5. The Fundamental Numerical Functor

In this section, we exhibit a projective generator of the category of numerical functors.

Theorem 12. — *Let K be a fixed module. The functor $\mathbf{B}[\operatorname{Hom}(K,-)]_n$, given by*

$$\begin{gathered}
M \mapsto \mathbf{B}[\operatorname{Hom}(K,M)]_n \\
[\chi \colon M \to N] \mapsto \begin{bmatrix} [\chi_*] \colon \mathbf{B}[\operatorname{Hom}(K,M)]_n \to \mathbf{B}[\operatorname{Hom}(K,N)]_n \\ [\alpha] \mapsto [\chi \circ \alpha] \end{bmatrix},
\end{gathered}$$

is numerical of degree n.

Proof. Since $\mathbf{B}[\mathrm{Hom}(K,-)]_n$ is the composition of $\mathbf{B}[-]_n$ with the Hom-functor, it suffices to prove $\mathbf{B}[-]_n$ is of degree n. Let $\chi_j\colon M \to N$ be homomorphisms, and let $x \in M$; then

$$[\chi_1 \diamond \cdots \diamond \chi_{n+1}]([x]) = [\chi_1(x) \diamond \cdots \diamond \chi_{n+1}(x)] = 0.$$

Moreover, if $a \in \mathbf{B}$ and $\chi\colon M \to N$, then

$$[a\chi]([x]) = [a\chi(x)] = \sum_{k=0}^{n} \binom{a}{k} \left[\mathop{\Diamond}_{k} \chi(x)\right] = \sum_{k=0}^{n} \binom{a}{k} \left[\mathop{\Diamond}_{k} \chi\right]([x]).$$

We infer that $\mathbf{B}[-]_n$ is numerical of degree n. □

Definition 13. — The functor

$$\mathbf{B}[\mathrm{Hom}(\mathbf{B}^n,-)]_n$$

will be called the **fundamental numerical functor** of degree n. ◇

§6. The Numerical Yoneda Correspondence

Theorem 13: The Numerical Yoneda Lemma. — *Let K be a fixed module, and F a numerical functor of degree n. The map*

$$\begin{aligned} \mathrm{Y}_{K,F}\colon \mathrm{Nat}\left(\mathbf{B}[\mathrm{Hom}(K,-)]_n, F\right) &\to F(K) \\ \eta &\mapsto \eta_K([1_K]) \end{aligned}$$

is an isomorphism of modules.

The isomorphism is natural, in the sense that the following two diagrams commute:

$$\begin{tikzcd} K \arrow[d, "\beta"'] & \mathrm{Nat}(\mathbf{B}[\mathrm{Hom}(K,-)]_n, F) \arrow[r, "\mathrm{Y}_{K,F}"] \arrow[d, "{[\beta^*]^*}"'] & F(K) \arrow[d, "F(\beta)"] \\ L & \mathrm{Nat}(\mathbf{B}[\mathrm{Hom}(L,-)]_n, F) \arrow[r, "\mathrm{Y}_{L,F}"'] & F(L) \end{tikzcd}$$

$$\begin{tikzcd} F \arrow[d, "\xi"'] & \mathrm{Nat}(\mathbf{B}[\mathrm{Hom}(K,-)]_n, F) \arrow[r, "\mathrm{Y}_{K,F}"] \arrow[d, "\xi_*"'] & F(K) \arrow[d, "\xi_K"] \\ G & \mathrm{Nat}(\mathbf{B}[\mathrm{Hom}(K,-)]_n, G) \arrow[r, "\mathrm{Y}_{K,G}"'] & G(K) \end{tikzcd}$$

Proof. The proof is the usual one. Consider the following commutative diagram:

$$\begin{tikzcd} K \arrow[d, "\alpha"'] & \mathbf{B}[\mathrm{Hom}(K,K)]_n \arrow[r, "\eta_K"] \arrow[d, "{[\alpha_*]}"'] & F(K) \arrow[d, "F(\alpha)"] & {[1_K]} \arrow[r] \arrow[d] & \eta_K([1_K]) \arrow[d] \\ M & \mathbf{B}[\mathrm{Hom}(K,M)]_n \arrow[r, "\eta_M"'] & F(M) & {[\alpha]} \arrow[r] & \eta_M([\alpha]) = F(\alpha)(\eta_K([1_K])) \end{tikzcd}$$

Upon inspection, we find that $\mathrm{Y}_{K,F}$ has the inverse

$$y \mapsto \begin{bmatrix} \eta_M \colon \mathbf{B}[\mathrm{Hom}(K,M)]_n \to F(M) \\ [\alpha] \mapsto F(\alpha)(y) \end{bmatrix}.$$

When defining this inverse, the numericality of F is used in an essential way to ensure that the map

$$\mathrm{Hom}(K,M) \to \mathrm{Hom}(F(K),F(M))$$

factor through $\mathbf{B}[\mathrm{Hom}(K,M)]_n$.

The naturality of Y is obvious. □

In particular, putting $F = \mathbf{B}[\mathrm{Hom}(K,-)]_n$, we obtain a module isomorphism

$$\mathrm{Nat}(\mathbf{B}[\mathrm{Hom}(K,-)]_n) \cong \mathbf{B}[\mathrm{Hom}(K,K)]_n = \mathbf{B}[\mathrm{End}\,K]_n,$$

given by the map

$$\mathrm{Y} \colon \eta \mapsto \eta_K([1_K])$$

with inverse

$$\mathrm{Y}^{-1} \colon [\sigma] \mapsto \begin{bmatrix} [\sigma^*] \colon \mathbf{B}[\mathrm{Hom}(K,-)]_n \to \mathbf{B}[\mathrm{Hom}(K,-)]_n \\ [\alpha] \mapsto [\alpha \circ \sigma]. \end{bmatrix}.$$

Recall from [10] (Definition 9) that the **augmentation algebras** of the module M are

$$\mathbf{B}[M]_n = \mathbf{B}[M]/I_n,$$

where $\mathbf{B}[M]$ denotes the free algebra under the **sum multiplication**

$$[x][y] = [x+y]$$

and I_n denotes the ideal

$$\begin{aligned} I_n = {} & (\, [x_1 \diamond \cdots \diamond x_{n+1}] \mid x_i \in M \,) \\ & + \left([rx] - \sum_{k=0}^{n} \binom{r}{k} \left[\underset{k}{\Diamond} x \right] \,\middle|\, r \in \mathbf{B},\ x \in M \right), \quad n \geqslant -1. \end{aligned}$$

When M is itself an algebra, there will be a corresponding **product multiplication** on $\mathbf{B}[M]$:

$$[x] \star [y] = [xy],$$

which descends unto the augmentation algebras $\mathbf{B}[M]_n$.

In the particular case given by the Yoneda correspondence above,

$$\begin{aligned} \mathrm{Y}^{-1}([\sigma] \star [\tau]) = \mathrm{Y}^{-1}([\sigma\tau]) &= [(\sigma\tau)^*] \\ &= [\tau^*] \circ [\sigma^*] = \mathrm{Y}^{-1}([\tau]) \circ \mathrm{Y}^{-1}([\sigma]). \end{aligned}$$

The product multiplication is reversed by Y, and we may thus conclude:

Theorem 14. — *The Yoneda correspondence provides an isomorphism of rings*

$$\left(\operatorname{Nat} \mathbf{B}[\operatorname{Hom}(K,-)]_n\right)^\circ \cong \mathbf{B}[\operatorname{End} K]_n,$$

where the former is equipped with composition, and the latter with the product multiplication.

§7. The Morita Equivalence

We proceed to demonstrate the equivalence of $\mathfrak{Num}_n$ with a suitable module category.

Lemma 2. — *A polynomial functor of degree n that vanishes on* $\mathbf{B}^n$ *is identically zero.*

Proof. Suppose that F is polynomial of degree n, and that $F(\mathbf{B}^n) = 0$. We shall shew that $F(\mathbf{B}^q) = 0$ for all natural numbers q.

Consider first the case $q \leqslant n$. Then $\mathbf{B}^q$ is a direct summand of $\mathbf{B}^n$, so $F(\mathbf{B}^q)$ is a direct summand of $F(\mathbf{B}^n) = 0$.

Proceeding by induction, suppose $F(\mathbf{B}^{q-1}) = 0$ for some $q-1 \geqslant n$. Decompose

$$1_{\mathbf{B}^q} = \pi_1 + \cdots + \pi_q,$$

where $\pi_j \colon \mathbf{B}^q \to \mathbf{B}^q$ denotes the jth projection. Since F is polynomial of degree n, and therefore of degree $q-1$,

$$0 = F(\pi_1 \diamond \cdots \diamond \pi_q) = \sum_{J \subseteq [q]} (-1)^{q-|J|} F\left(\sum_J \pi_j\right).$$

Consider a J with $|J| \leqslant q-1$. Since $\sum_J \pi_j$ factors through $\mathbf{B}^{q-1}$, the homomorphism $F\left(\sum_J \pi_j\right)$ factors through $F(\mathbf{B}^{q-1}) = 0$. Only $J = [q]$ will give a non-trivial contribution to the sum above, yielding

$$0 = F(\pi_1 + \cdots + \pi_q) = F(1_{\mathbf{B}^q}) = 1_{F(\mathbf{B}^q)};$$

hence $F(\mathbf{B}^q) = 0$. □

The next result was established by Pirashvili for polynomial functors in [6].

Theorem 15. — *The fundamental numerical functor*

$$\mathbf{B}[\operatorname{Hom}(\mathbf{B}^n,-)]_n$$

is a small projective generator for $\mathfrak{Num}_n$*, through which there is a Morita equivalence*

$$\mathfrak{Num}_n \sim {}_{\mathbf{B}[\mathbf{B}^{n\times n}]_n}\mathfrak{Mod},$$

where $\mathbf{B}[\mathbf{B}^{n\times n}]_n$ *carries the product multiplication.*

More precisely, the functor F corresponds to the abelian group $F(\mathbf{B}^n)$, with module structure given by the equation

$$[\tau] \cdot x = F(\tau)(x).$$

Proof. Step 1: $\mathbf{B}[\mathrm{Hom}(\mathbf{B}^n, -)]_n$ *is projective.* We must shew that

$$\mathrm{Nat}(\mathbf{B}[\mathrm{Hom}(\mathbf{B}^n, -)]_n, -)$$

is right-exact, or preserves epimorphisms. Hence let $\eta\colon F \to G$ be epimorphic, so that each η_M is onto. The following diagram, constructed by aid of the Yoneda Lemma, shews that η_* is epimorphic:

$$\begin{tikzcd}
\mathrm{Nat}(\mathbf{B}[\mathrm{Hom}(\mathbf{B}^n, -)]_n, F) \arrow[r, leftrightarrow, "\mathrm{Y}"] \arrow[d, "\eta_*"'] & F(\mathbf{B}^n) \arrow[d, "\eta_{\mathbf{B}^n}"] \\
\mathrm{Nat}(\mathbf{B}[\mathrm{Hom}(\mathbf{B}^n, -)]_n, G) \arrow[r, leftrightarrow, "\mathrm{Y}"'] & G(\mathbf{B}^n)
\end{tikzcd}$$

Step 2: $\mathbf{B}[\mathrm{Hom}(\mathbf{B}^n, -)]_n$ *is a generator.* By the lemma,

$$0 = \mathrm{Nat}(\mathbf{B}[\mathrm{Hom}(\mathbf{B}^n, -)]_n, F) \cong F(\mathbf{B}^n)$$

implies $F = 0$.

Step 3: $\mathbf{B}[\mathrm{Hom}(\mathbf{B}^n, -)]_n$ *is small.* Compute, using the Yoneda Lemma:

$$\begin{aligned}
\mathrm{Nat}\left(\mathbf{B}[\mathrm{Hom}(\mathbf{B}^n, -)]_n, \bigoplus F_k\right) &\cong \left(\bigoplus F_k\right)(\mathbf{B}^n) = \bigoplus F_k(\mathbf{B}^n) \\
&\cong \bigoplus \mathrm{Nat}\left(\mathbf{B}[\mathrm{Hom}(\mathbf{B}^n, -)]_n, F_k\right).
\end{aligned}$$

Step 4: The Morita equivalence. As $\mathfrak{Num}_n$ is an abelian category with arbitrary direct sums, there is a Morita equivalence:

$$\begin{tikzcd}
\mathfrak{Num}_n \arrow[r, bend left, "\mathrm{Nat}(\mathbf{B}[\mathrm{Hom}(\mathbf{B}^n,-)]_n,-)"] & {}_S\mathfrak{Mod} \arrow[l, bend left, "\mathbf{B}[\mathrm{Hom}(\mathbf{B}^n,-)]_n \otimes_S -"]
\end{tikzcd}$$

The new base ring is

$$S = \left(\mathrm{Nat}\,\mathbf{B}[\mathrm{Hom}(\mathbf{B}^n, -)]_n\right)^\circ \cong \mathbf{B}[\mathrm{End}\,\mathbf{B}^n]_n = \mathbf{B}[\mathbf{B}^{n\times n}]_n.$$

Plainly, the functor F corresponds to the abelian group

$$\mathrm{Nat}(\mathbf{B}[\mathrm{Hom}(\mathbf{B}^n, -)]_n, F) \cong F(\mathbf{B}^n).$$

Step 5: The module structure. Under the Yoneda map, an element $x \in F(\mathbf{B}^n)$ will correspond to the natural transformation

$$\eta_M\colon \mathbf{B}[\mathrm{Hom}(\mathbf{B}^n, M)]_n \to F(M)$$

$$[\alpha] \mapsto F(\alpha)(x).$$

Likewise, a scalar $[\tau] \in \mathbf{B}[\mathbf{B}^{n\times n}]_n$ will correspond to

$$\begin{aligned} \sigma_M \colon \mathbf{B}[\mathrm{Hom}(\mathbf{B}^n, M)]_n &\to \mathbf{B}[\mathrm{Hom}(\mathbf{B}^n, M)]_n \\ [\alpha] &\mapsto [\alpha \circ \tau]. \end{aligned}$$

The product of the scalar σ and the module element η is the transformation

$$\begin{aligned} (\eta \circ \sigma)_M \colon \mathbf{B}[\mathrm{Hom}(\mathbf{B}^n, M)]_n &\to F(M) \\ [\alpha] &\mapsto F(\alpha \circ \tau)(x), \end{aligned}$$

which under the Yoneda map corresponds to

$$(\eta \circ \sigma)_{\mathbf{B}^n}([\mathrm{I}_{\mathbf{B}^n}]) = F(\mathrm{I}_{\mathbf{B}^n} \circ \tau)(x) = F(\tau)(x) \in F(\mathbf{B}^n).$$

The scalar multiplication on $F(\mathbf{B}^n)$ is thus given by the formula

$$[\tau] \cdot x = F(\tau)(x),$$

and the proof is finished. □

§8. The Divided Power Map

A key rôle in the theory of polynomial functors is played by the rings $\mathbf{B}[\mathbf{B}^{n\times n}]_n$ and $\Gamma^n(\mathbf{B}^{n\times n})$, in that their modules encode numerical and homogeneous functors (of degree n), respectively. One manifest way of relating the two species of functors will then be to exhibit homomorphisms between the respective rings. This shall form the topic of the present section.

Let us begin with somewhat greater generality. Taking as our starting-point a module M, we propose a study of the **divided power map**

$$\begin{aligned} \gamma_n \colon M &\to \Gamma^n(M) \\ x &\mapsto x^{[n]}. \end{aligned}$$

Theorem 16. — *The divided power map is numerical of degree n and therefore induces a linear map*

$$\begin{aligned} \gamma_n \colon \mathbf{B}[M]_n &\to \Gamma^n(M) \\ [x] &\mapsto x^{[n]}. \end{aligned}$$

This is a natural transformation of (numerical) functors.

Proof. Since γ_n is homogeneous of degree n, it is also numerical of the same degree. □

Lemma 3. — *If $x_1, \ldots, x_n \in M$, then*

$$(x_1 \diamond \cdots \diamond x_n)^{[n]} = x_1 \cdots x_n.$$

Proof. By the definition of deviations,

$$(x_1 \diamond \cdots \diamond x_n)^{[n]} = \sum_{I \subseteq [n]} (-1)^{n-|I|} \left(\sum_{i \in I} x_i \right)^{[n]}.$$

A given monomial $x^{[X]}$ will occur in those terms for which $\#X \subseteq I$. Its coëfficient will be

$$\sum_{\#X \subseteq I \subseteq [n]} (-1)^{n-|I|},$$

which is 1 if $\#X = [n]$ and 0 otherwise. □

Theorem 17. — *Let M be finitely generated and free. The co-kernel of the homomorphism*

$$(\pi, \gamma_n) \colon \mathbf{B}[M]_n \to \mathbf{B}[M]_{n-1} \oplus \Gamma^n(M)$$

is

$$\operatorname{Coker}(\pi, \gamma_n) \cong \Gamma^n(M) / \langle x_1 \cdots x_n \mid x_i \in M \rangle.$$

In particular, (π, γ_n) is an injection of finite index.

Proof. Let $\{e_1, \ldots, e_k\}$ be a basis for M. Then the elements

$$[f_1 \diamond \cdots \diamond f_m], \quad f_i \in \{e_1, \ldots, e_k\},$$

for $0 \leqslant m \leqslant n$, constitute a basis for $\mathbf{B}[M]_n$. The image of (π, γ_n) is generated by the images

$$(\pi, \gamma_n)([f_1 \diamond \cdots \diamond f_m]) = \left([f_1 \diamond \cdots \diamond f_m], [f_1 \diamond \cdots \diamond f_m]^{[n]} \right), \quad 0 \leqslant m < n;$$

and

$$(\pi, \gamma_n)([f_1 \diamond \cdots \diamond f_n]) = \left(0, (f_1 \diamond \cdots \diamond f_n)^{[n]} \right) = (0, f_1 \cdots f_n).$$

The relations

$$[f_1 \diamond \cdots \diamond f_m] \equiv -[f_1 \diamond \cdots \diamond f_m]^{[n]} \mod \operatorname{Im}(\pi, \gamma_n)$$

permit the representation of each element of the co-kernel by a sum of divided nth powers, while the relations

$$f_1 \cdots f_n \equiv 0 \mod \operatorname{Im}(\pi, \gamma_n)$$

yield the desired factor module of $\Gamma^n(M)$. □

Theorem 18. — *Let M be finitely generated and free. The kernel of the homomorphism*

$$\gamma_n \colon \mathbf{B}[M]_n \to \Gamma^n(M)$$

is

$$\operatorname{Ker} \gamma_n = \mathbf{Q} \otimes_{\mathbf{Z}} \langle [rz] - r^n [z] \mid r \in \mathbf{B},\ z \in M \rangle \cap \mathbf{B}[M]_n.$$

Proof. Let $\{e_1, \dots, e_k\}$ be a basis for M. Then, as m ranges from 0 to n, the elements

$$[f_1 \diamond \cdots \diamond f_m], \quad f_i \in \{e_1, \dots, e_k\},$$

will constitute a basis for $\mathbf{B}[M]_n$.

Denote

$$L = \mathbf{Q} \otimes_{\mathbf{Z}} \langle [rz] - r^n[z] \mid r \in \mathbf{B}, z \in M \rangle;$$

then evidently

$$L \cap \mathbf{B}[M]_n \subseteq \operatorname{Ker} \gamma_n.$$

We now shew the reverse inclusion.

Calculating modulo L, we have, for any z,

$$\begin{aligned} \left[\underset{m}{\Diamond} z\right] &= \sum_{I \subseteq [m]} (-1)^{m-|I|} \left[\sum_{i \in I} z\right] = \sum_{r=0}^{m} (-1)^{m-r} \binom{m}{r} [rz] \\ &\equiv \sum_{r=0}^{m} (-1)^{m-r} \binom{m}{r} r^n [z] = m! \left\{ {n \atop m} \right\} [z], \end{aligned} \tag{1}$$

where $\left\{ {n \atop m} \right\}$ denotes a Stirling number of the second kind.

We may then write

$$\begin{aligned} m! \left\{ {n \atop m} \right\} [f_1 \diamond \cdots \diamond f_m] &= m! \left\{ {n \atop m} \right\} \sum_{I \subseteq [m]} (-1)^{m-|I|} \left[\sum_{i \in I} f_i\right] \\ &= \sum_{I \subseteq [m]} (-1)^{m-|I|} \left[\underset{m}{\Diamond} \sum_{i \in I} f_i\right] = \xi + \xi', \end{aligned}$$

where ξ is a sum of mth deviations, and ξ' collects the higher-order deviations. We calculate ξ:

$$\begin{aligned} \xi &= \sum_{I \subseteq [m]} (-1)^{m-|I|} \sum_{\substack{\# A \subseteq I \\ |A| = m}} \binom{m}{A} \left[\underset{a \in A}{\Diamond} f_a\right] \\ &= \sum_{\substack{\# A \subseteq [m] \\ |A| = m}} \left(\sum_{\# A \subseteq I \subseteq [m]} (-1)^{m-|I|} \right) \binom{m}{A} \left[\underset{a \in A}{\Diamond} f_a\right] \\ &= \sum_{\substack{\# A = [m] \\ |A| = m}} \binom{m}{A} \left[\underset{a \in A}{\Diamond} f_a\right] = \binom{m}{1, \dots, 1} [f_1 \diamond \cdots \diamond f_m] = m![f_1 \diamond \cdots \diamond f_m]. \end{aligned}$$

We thus have

$$m! \left\{ {n \atop m} \right\} [f_1 \diamond \cdots \diamond f_m] \equiv m![f_1 \diamond \cdots \diamond f_m] + \xi' \mod L,$$

and, consequently, provided $1 < m < n$ (so that $\left\{ {n \atop m} \right\} > 1$),

$$[f_1 \diamond \cdots \diamond f_m] \equiv \frac{1}{m!\left(\left\{ {n \atop m} \right\} - 1\right)} \cdot \xi' \mod L.$$

Now suppose $\omega \in \operatorname{Ker} \gamma_n$. Using the above relation, together with

$$[\diamond] = [0] \equiv 0 \mod L$$

and

$$[\diamond f] = [f] - [0] \equiv \frac{1}{n!\left\{ {n \atop n} \right\}} \left[\underset{n}{\Diamond} f \right] = \frac{1}{n!} \left[\underset{n}{\Diamond} f \right] \mod L,$$

which are both consequences of (1); we may express ω as a (fractional) linear combination of nth deviations of the basis elements e_i:

$$\omega \equiv \sum_{\substack{\#A \subseteq [k] \\ |A| = n}} c_A \left[\underset{a \in A}{\Diamond} e_a \right] \mod L.$$

Apply γ_n:

$$0 = \gamma_n(\omega) = \sum_{\substack{\#A \subseteq [k] \\ |A| = n}} c_A \gamma_n \left(\underset{a \in A}{\Diamond} e_a \right) = \sum_{\substack{\#A \subseteq [k] \\ |A| = n}} c_A e^A.$$

Because the elements $e^{[A]}$ constitute a basis for $\Gamma^n(M)$, it must be that all coëfficients $c_A = 0$, and hence $\omega \in L$. The proof is finished. □

Due to lack of torsion in the base ring, the divided power map may always be extended to a map

$$\gamma_n \colon \mathbf{Q} \otimes_{\mathbf{Z}} \mathbf{B}[M]_n \to \mathbf{Q} \otimes_{\mathbf{Z}} \Gamma^n(M).$$

This map (almost trivially) possesses an inverse. We shall, however, be interested in obtaining an inverse under a slightly less generous localisation.

Theorem 19. — *The homomorphism*

$$\begin{aligned} \varepsilon_n \colon \Gamma^n(M) &\to \mathbf{Q} \otimes_{\mathbf{Z}} \mathbf{B}[M]_n \\ x^{[A]} &\mapsto \frac{1}{\deg A} \left[\underset{A}{\Diamond} x \right] \end{aligned}$$

constitutes a section of the divided power map:

$$\gamma_n \varepsilon_n = 1_{\Gamma^n(M)}.$$

This leads to a direct sum decomposition:

$$\operatorname{Im} \varepsilon_n \cong \Gamma^n(M) \oplus (\operatorname{Ker} \gamma_n \cap \operatorname{Im} \varepsilon_n).$$

Proof. The relation $\gamma_n \varepsilon_n = 1$ is immediate, and then the following exact sequence splits:

$$0 \longrightarrow \operatorname{Ker} \gamma_n \cap \operatorname{Im} \varepsilon_n \longrightarrow \operatorname{Im} \varepsilon_n \underset{\varepsilon_n}{\overset{\gamma_n}{\rightleftarrows}} \Gamma^n(M) \longrightarrow 0$$

□

It will now be appropriate to specialise the preceding discussion to the particular rings $\mathbf{B}[\mathbf{B}^{n\times n}]_n$ and $\Gamma^n(\mathbf{B}^{n\times n})$. Recall that both rings are equipped with a product multiplication, given by the laws

$$[\alpha] \star [\beta] = [\alpha\beta], \qquad \alpha^{[n]} \star \beta^{[n]} = (\alpha\beta)^{[n]}.$$

Theorem 20. — *The maps*

$$\gamma_n \colon \mathbf{B}[\mathbf{B}^{n\times n}]_n \to \Gamma^n(\mathbf{B}^{n\times n}), \qquad \varepsilon_n \colon \Gamma^n(\mathbf{B}^{n\times n}) \to \mathbf{Q} \otimes_{\mathbf{Z}} \mathbf{B}[\mathbf{B}^{n\times n}]_n$$

are homomorphisms of algebras, when both rings are equipped with the product multiplication.

Proof. Calculate:

$$\gamma_n([\alpha]) \star \gamma_n([\beta]) = \alpha^{[n]} \star \beta^{[n]} = (\alpha\beta)^{[n]} = \gamma_n([\alpha\beta]) = \gamma([\alpha] \star [\beta]).$$

To shew ε_n preserves multiplication, it will be enough to consider pure powers $\alpha^{[n]}$ and $\beta^{[n]}$. The relation

$$\left[\underset{n}{\Diamond}\, \alpha\right] \star \left[\underset{n}{\Diamond}\, \beta\right] = n! \left[\underset{n}{\Diamond}\, \alpha\beta\right]$$

is readily verified, by means of simple algebraical manipulations.[5] We may then compute:

$$\begin{aligned} \varepsilon_n(\alpha^{[n]}) \star \varepsilon_n(\beta^{[n]}) &= \frac{1}{n!}\left[\underset{n}{\Diamond}\, \alpha\right] \star \frac{1}{n!}\left[\underset{n}{\Diamond}\, \beta\right] \\ &= \frac{1}{n!}\left[\underset{n}{\Diamond}\, \alpha\beta\right] = \varepsilon_n((\alpha\beta)^{[n]}) = \varepsilon_n(\alpha^{[n]} \star \beta^{[n]}). \end{aligned}$$

□

Finally, let us put forth the module-theoretic equivalent of quasi-homogeneity.

Theorem 21. — *Let F be a numerical functor, corresponding to the $\mathbf{B}[\mathbf{B}^{n\times n}]_n$-module M. The functor F is quasi-homogeneous of degree n if and only if M is a module over*

$$\operatorname{Im} \gamma_n \cong \mathbf{B}[\mathbf{B}^{n\times n}]_n / \operatorname{Ker} \gamma_n.$$

[5] Compare the Deviation Formula, Theorem 7.1, in [9].

Proof. Recall that the scalar multiplication of $\mathbf{B}[\mathbf{B}^{n\times n}]_n$ on $M = F(\mathbf{B}^n)$ is given by

$$[\sigma]x = F(\sigma)(x), \quad \sigma \in \mathbf{B}^{n\times n},\ x \in F(\mathbf{B}^n).$$

The requirement that $\operatorname{Ker}\gamma_n$ annihilate $F(\mathbf{B}^n)$ is equivalent to demanding that F itself vanish on

$$\operatorname{Ker}\gamma_n = \mathbf{Q}\otimes_{\mathbf{Z}} \left\langle [r\sigma] - r^n[\sigma] \,\middle|\, r \in \mathbf{B},\ \sigma \in \mathbf{B}^{n\times n}\right\rangle \cap \mathbf{B}[\mathbf{B}^{n\times n}]_n,$$

which would clearly be a consequence of quasi-homogeneity.

To shew that, conversely, quasi-homogeneity is implied by the equation

$$F(r\sigma) = r^n F(\sigma), \quad \sigma \in \mathbf{B}^{n\times n},$$

we first shew that

$$F(r\cdot \mathrm{I}_{\mathbf{B}^q}) = r^n F(\mathrm{I}_{\mathbf{B}^q})$$

for all natural numbers q. This is clear when $q \leqslant n$, for then $\mathrm{I}_{\mathbf{B}^q}$ factors through $\mathrm{I}_{\mathbf{B}^n}$. When $q > n$, split up into the canonical projections, and use induction:

$$\begin{aligned} F(r\cdot \mathrm{I}_{\mathbf{B}^q}) &= F(r\pi_1 + \cdots + r\pi_q) \\ &= -\sum_{I\subset[q]} (-1)^{q-|I|} F\left(\sum_{i\in I} r\pi_i\right) \\ &= -\sum_{I\subset[q]} (-1)^{q-|I|} r^n F\left(\sum_{i\in I} \pi_i\right) \\ &= r^n F(\pi_1 + \cdots + \pi_q) = r^n F(\mathrm{I}_{\mathbf{B}^q}). \end{aligned}$$

Finally, for an arbitrary homomorphism $\alpha\colon \mathbf{B}^p \to \mathbf{B}^q$, we have

$$F(r\alpha) = F(r\cdot \mathrm{I}_{\mathbf{B}^q})F(\alpha) = r^n F(\mathrm{I}_{\mathbf{B}^q})F(\alpha) = r^n F(\alpha).$$

□

§9. Restriction and Extension of Scalars

The divided power map

$$\gamma_n\colon \mathbf{B}[\mathbf{B}^{n\times n}]_n \to \Gamma^n(\mathbf{B}^{n\times n})$$

gives rise to two natural functors between the corresponding module categories, viz. restriction and extension of scalars. We consider them in turn.

Restriction of scalars is the functor

$${}_{\Gamma^n(\mathbf{B}^{n\times n})}\mathfrak{Mod} \to {}_{\mathbf{B}[\mathbf{B}^{n\times n}]_n}\mathfrak{Mod},$$

which takes a $\Gamma^n(\mathbf{B}^{n\times n})$-module M and views it as a $\mathbf{B}[\mathbf{B}^{n\times n}]_n$-module under the multiplication

$$[\sigma]x = \gamma_n(\sigma)x = \sigma^{[n]}x.$$

On the functorial level, this corresponds to the forgetful functor

$$\mathfrak{Hom}_n \to \mathfrak{Num}_n.$$

Extension of scalars is the functor

$${}_{\mathbf{B}[\mathbf{B}^{n\times n}]_n}\mathfrak{Mod} \to {}_{\Gamma^n(\mathbf{B}^{n\times n})}\mathfrak{Mod},$$

which takes a $\mathbf{B}[\mathbf{B}^{n\times n}]_n$-module M and transforms it into a $\Gamma^n(\mathbf{B}^{n\times n})$-module

$$\Gamma^n(\mathbf{B}^{n\times n}) \otimes_{\mathbf{B}[\mathbf{B}^{n\times n}]_n} M$$

through the tensor product.

Let us examine its action on the functorial level. To this end, denote by

$$\begin{aligned} P &= \mathbf{B}[\mathrm{Hom}(\mathbf{B}^n, -)]_n \\ Q &= \Gamma^n \mathrm{Hom}(\mathbf{B}^n, -) \end{aligned}$$

the projective generators of the categories $\mathfrak{Num}_n$ and $\mathfrak{Hom}_n$, respectively. A functor $F \in \mathfrak{Num}_n$ will then correspond to the $\mathbf{B}[\mathbf{B}^{n\times n}]_n$-module

$$M = \mathrm{Nat}(P, F).$$

Extension of scalars transforms it into the $\Gamma^n(\mathbf{B}^{n\times n})$-module

$$N = \Gamma^n(\mathbf{B}^{n\times n}) \otimes_{\mathbf{B}[\mathbf{B}^{n\times n}]_n} M = \Gamma^n(\mathbf{B}^{n\times n}) \otimes_{\mathbf{B}[\mathbf{B}^{n\times n}]_n} \mathrm{Nat}(P, F),$$

which corresponds to the homogeneous functor

$$\begin{aligned} G &= Q \otimes_{\Gamma^n(\mathbf{B}^{n\times n})} N \\ &= Q \otimes_{\Gamma^n(\mathbf{B}^{n\times n})} \Gamma^n(\mathbf{B}^{n\times n}) \otimes_{\mathbf{B}[\mathbf{B}^{n\times n}]_n} \mathrm{Nat}(P, F) \\ &= Q \otimes_{\mathbf{B}[\mathbf{B}^{n\times n}]_n} \mathrm{Nat}(P, F). \end{aligned}$$

This tensor product is interpreted in the usual way. By definition,

$$\begin{aligned} P &\mapsto Q \otimes_{\mathbf{B}[\mathbf{B}^{n\times n}]_n} \mathrm{Nat}(P, P) \\ &= Q \otimes_{\mathbf{B}[\mathbf{B}^{n\times n}]_n} \mathbf{B}[\mathbf{B}^{n\times n}]_n^\circ = Q, \end{aligned}$$

and then extension is performed by means of direct sums and right-exactness.

We summarise in a theorem.

Theorem 22. — *Consider the divided power map*

$$\gamma_n \colon \mathbf{B}[\mathbf{B}^{n\times n}]_n \to \Gamma^n(\mathbf{B}^{n\times n}).$$

- *Restriction of scalars*

$${}_{\Gamma^n(\mathbf{B}^{n\times n})}\mathfrak{Mod} \to {}_{\mathbf{B}[\mathbf{B}^{n\times n}]_n}\mathfrak{Mod}$$

  corresponds to the forgetful functor

$$\mathfrak{Hom}_n \to \mathfrak{Num}_n.$$

- *Extension of scalars*
$$_{\mathbf{B}[\mathbf{B}^{n\times n}]_n}\mathfrak{Mod} \to {}_{\Gamma^n(\mathbf{B}^{n\times n})}\mathfrak{Mod}$$
corresponds to the functor
$$\mathfrak{Num}_n \to \mathfrak{Hom}_n$$
which maps
$$\mathbf{B}[\mathrm{Hom}(\mathbf{B}^n, -)]_n \mapsto \Gamma^n \mathrm{Hom}(\mathbf{B}^n, -),$$
extended by direct sums and right-exactness.

§10. Numerical versus Strict Polynomial Functors

Lastly, we present one half of the Polynomial Functor Theorem, Theorem 11.3 of [9], recording a necessary and sufficient condition for a numerical functor to be strict polynomial. (Whereas the present condition is stated in terms of the description of functors as modules, the other half proposes a combinatorial criterion.)

Theorem 23: The Polynomial Functor Theorem. — *Let F be a quasi-homogeneous functor of degree n, corresponding to the $\mathbf{B}[\mathbf{B}^{n\times n}]_n$-module M and the $\mathrm{Im}\,\gamma_n$-module N. The following constructs are equivalent:*

A. *Imposing the structure of homogeneous functor, of degree n, upon F.*

B. *Giving M the structure of $\mathrm{Im}\,\varepsilon_n$-module.*

C. *Giving N the structure of $\Gamma^n(\mathbf{B}^{n\times n})$-module.*

Proof. The equivalence of A and C is immediate. From the isomorphism
$$\mathrm{Im}\,\varepsilon_n \cong \Gamma^n(\mathbf{B}^{n\times n}) \times (\mathrm{Ker}\,\gamma_n \cap \mathrm{Im}\,\varepsilon_n),$$
we conclude that $\Gamma^n(\mathbf{B}^{n\times n})$-modules canonically correspond to $\mathrm{Im}\,\varepsilon_n$-modules, and vice versa. (The ring $\mathrm{Ker}\,\gamma_n \cap \mathrm{Im}\,\varepsilon_n$ corresponds to subfunctors of lower degree. By considering quasi-homogeneous functors only, modules over this ring will be zero.) This demonstrates the equivalence of B and C. □

Once again, we caution the reader that, even in the case M may be considered an $\mathrm{Im}\,\varepsilon_n$-module, this module structure will not be unique. There are in general many strict polynomial structures on the same functor, even of different degrees.

Example 7. — In the *affine* case (degree 0 and 1), numerical and strict polynomial functors coincide. This is no longer the case in higher degrees. Already in the quadratic case, there exist numerical functors which do not admit a strict polynomial structure.

Yet, some concordance will be retained in the quadratic case, in that any *quasi-homogeneous* functor may be given a unique strict polynomial structure, which makes it homogeneous of degree 2. This stems from the map

$$\gamma_2 \colon \mathbf{B}[\mathbf{B}^{2\times 2}]_2 \to \Gamma^2(\mathbf{B}^{2\times 2})$$

being onto, so that there is, in fact, a split exact sequence

$$\begin{tikzcd} 0 \arrow[r] & \operatorname{Ker}\gamma_2 \arrow[r] & \mathbf{B}[\mathbf{B}^{2\times 2}]_2 \arrow[r, shift left, "\gamma_2"] & \Gamma^2(\mathbf{B}^{2\times 2}) \arrow[l, dotted, shift left, "\varepsilon_2"] \arrow[r] & 0 \end{tikzcd}$$

△

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